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The effect of surface tension on the contraction coefficient of a jet

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Abstract
Two-dimensional free surface potential flow issued from an opening of a container is considered. The flow is assumed to be inviscid and incompressible. The mathematical problem, which is characterized by the nonlinear boundary condition on the free surface of an unknown equation, is solved via a series truncation. We computed solutions for all Weber numbers. Our problem is an extension of the work done by Ackerberg and Liu (1987 *Phys. Fluids* **30** 289–96), the results confirm and extend their results.

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1. Introduction

We consider the steady two-dimensional motion of a fluid flowing through a slot of width $2L$ with inclined walls (see figure 1). The fluid is assumed to be inviscid, incompressible and the flow is irrotational. We neglect the effect of gravity but we take into account the surface tension. Far downstream the velocity approaches a constant U and the depth of the fluid is $2H$. Far upstream the fluid is at rest.

As we shall see, the flow is characterized by three parameters, the inclination angle β (see figure 1), the angle at the separation point between the wall and the free streamlines γ and the Weber number α defined by

$$\alpha = \frac{\rho U^2 L}{T} \tag{1}$$

where T is the surface tension and ρ is the density of the fluid.

When the effects of surface tension and gravity g are neglected, the classical exact solution can be found via the hodograph transformation (see, for example, [2, 3] for $\beta = \frac{\pi}{2}$).

If the effects of surface tension or gravity are considered, the boundary condition at the free surface, known as the Bernoulli equation, is nonlinear and no exact analytical solution is known. Different combinations and some varieties of this problem have been considered.

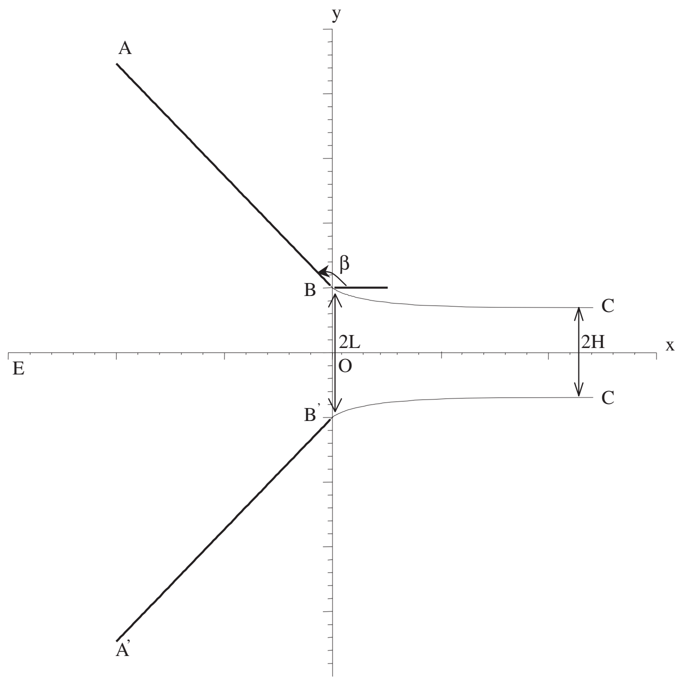


Figure 1. Sketch of the flow and of the coordinates. The width of the opening is $2L$ and the depth of the flow at infinity is $2H$. The x axis is along the streamline EOC and the y axis forms the angle $\beta - \frac{\pi}{2}$ with the walls AB and $A'B'$. The figure is an actual computed surface profile for $\beta = \pi/3$ and the Weber number $\alpha = 100$.

Neglecting the surface tension and considering the effect of gravity leads to the assumption that the depth far upstream and the velocity are constant. Some variants of this problem are: flow over a dump, flow under a sluice gate, flow over an obstacle, etc. Many authors have investigated these problems: Tuck [4], Vanden-Breock [6, 7], Lee and Vanden-Breock [5].

Ackerberg and Liu [1] have considered the problem neglecting gravity and considering the effect of surface tension. They solved the problem via a finite difference method and the mesh points were throughout the fluid domain. They computed solutions for all Weber numbers $\alpha \geqslant \tilde{\alpha} = 6.801\,483$ but they were unable to compute a solution for $\alpha < \tilde{\alpha} = 6.801\,483$. In this paper, we solve the fully nonlinear problem numerically, and the mesh points are only on the free surface. For each angle β, we found that there is a value $\alpha^* < \tilde{\alpha}$ such that there exists a unique solution, for all $\alpha \geqslant \alpha^*$. If $\alpha < \alpha^*$ the numerical scheme diverges. Our results confirm and extend the results of Ackerberg and Liu [1].

The problem is formulated in section 2, the numerical procedure is described in section 3 and we conclude this work by a discussion of the results in section 4.

2. Formulation of the problem

Let us consider the motion of a two-dimensional flow of a fluid contained between two semi-infinite inclined walls (figure 1). The opening between the walls is of width $2L$. The flow is considered to be inviscid, incompressible and steady. In the absence of gravity, the main flow stays horizontal and extends to infinity between the two free surfaces (figure 1). Due to symmetry, the straight line of symmetry EOC is a streamline. We introduce Cartesian

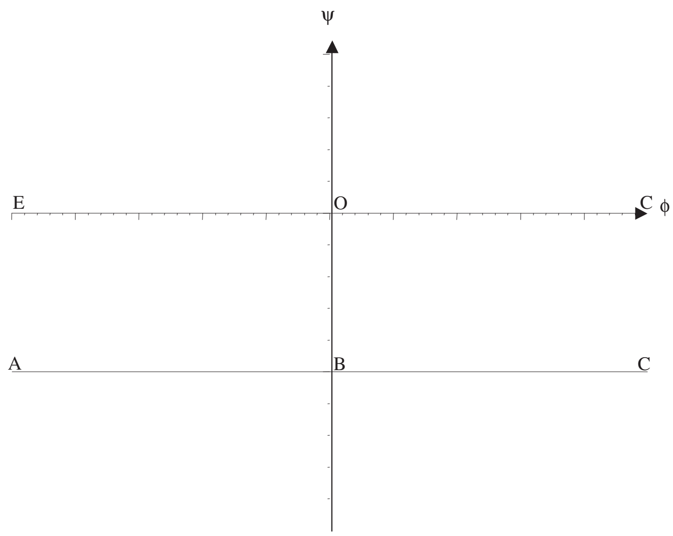


Figure 2. The complex potential f plane.

coordinates with the streamline EOC on the x axis and the y axis is vertically along the opening BB'. Far upstream the velocity approaches zero. Far downstream, we assume that the velocity approaches a constant U and the depth of the fluid tends to a constant $2H$. Because of the symmetry of the flow field, we need only consider the upper half of the flow region which is contained between the x axis and the streamline ABC.

We define dimensionless variables by taking U as the unit velocity and L as the unit length. We introduce the potential function ϕ and the stream function ψ. Without loss of generality we choose $\phi = 0$ at $(x, y) = (0, 1)$ and $\psi = 0$ on the stream line ABC. It follows from the choice of the dimensionless variables that $\psi = -1$ on the stream line EOC. In order to use the conformal mapping techniques, we consider the flow in the complex plane $z = x + \mathrm{i}y$ and the complex potential function $f = \phi + \mathrm{i}\psi$. The upper half of the flow region in the z plane will be mapped via the potential function f onto the semi-infinite strip $(-\infty < \phi < \infty, -1 < \psi < 0)$ (figure 2).

We introduce the complex velocity $\zeta = u - \mathrm{i}v = C\frac{\mathrm{d}f}{\mathrm{d}z}$. Here $C = \frac{H}{L}$ is the contraction coefficient.

On the free surface, the atmospheric pressure P_0 is constant, hence the Bernoulli equation yields

$$\bar{P} + \tfrac{1}{2}\rho\bar{q}^2 = P_0 + \tfrac{1}{2}\rho U^2 \qquad \text{on} \quad \psi = 0 \quad \phi > 0 \tag{2}$$

where $\bar{P}$ and $\bar{q}$ are the fluid pressure and the speed just inside the free surface, respectively. The right-hand side of equation (2) is evaluated from the condition far downstream.

A relationship between $\bar{P}$ and P_0 is given by Laplace's capillarity formula

$$\bar{P} - P_0 = T\bar{K}. \tag{3}$$

Here $\bar{K}$ is the curvature of the free surface and T is the surface tension.

If we substitute (3) in (2) we obtain

$$\frac{1}{2}\bar{q}^2 - \frac{T}{\rho}\bar{K} = \frac{1}{2}U^2. \tag{4}$$

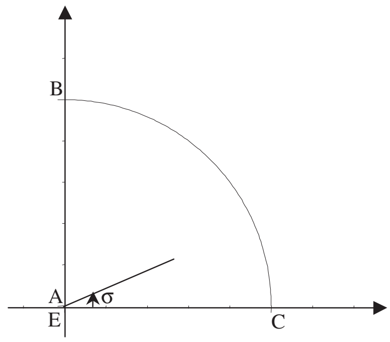


Figure 3. The complex potential t plane.

In dimensionless variables (4) becomes

$$\frac{1}{2}q^2 - \frac{1}{\alpha}K = \frac{1}{2} \tag{5}$$

where α is the Weber number defined by (1).

In order that the curvature be well defined we introduce the function $\tau - \mathrm{i}\theta$ as

$$\zeta = u - \mathrm{i}v = \mathrm{e}^{\tau - \mathrm{i}\theta}. \tag{6}$$

In these new variables (5) becomes

$$\frac{\partial\theta}{\partial\phi} = \frac{2}{\alpha}(\mathrm{e}^{\tau} - \mathrm{e}^{-\tau}) \qquad 0 < \phi < \infty \tag{7}$$

The kinematic condition on AB and EOC can be expressed as

$$\mathrm{Re}\,\zeta/\mathrm{Im}\zeta = -\cot\beta \qquad \text{on} \quad \psi = 0 \quad -\infty < \phi < 0. \tag{8}$$

$$\mathrm{Im}\,\zeta = 0 \qquad \text{on} \quad \psi = -1 \quad -\infty < \phi < \infty. \tag{9}$$

We shall seek $\tau - \mathrm{i}\theta$ as an analytic function of $f = \phi + \mathrm{i}\psi$ in the strip $-1 < \psi < 0$, satisfying the conditions (7), (8) and (9).

3. Numerical procedure

We define a new variable t by the relation

$$f = \frac{2}{\pi}\log\left(\frac{-2\mathrm{i}t}{1 - t^2}\right). \tag{10}$$

This transformation maps the flow domain into the first quarter of the unit disc in the complex t plane. The problem in the complex t plane is illustrated in figure 3.

3.1. Local behaviour of ζ at $t = 0, t = i$

The flow is potential everywhere except at $t = 0$ and $t = i$, where we have a flow around a corner at $t = i$ and stagnant fluid at $t = 0$. Hence a local analysis is required.

3.1.1. Asymptotic behaviour far upstream ($t = 0$). Far upstream the opening BO will appear as a line sink, hence the flow is characterized by the complex potential function in the z plane as

$$f \sim \frac{1}{\pi - \beta}\log(z) \qquad \text{if} \quad |z| \to +\infty \quad \text{and} \quad \arg z > \beta.$$

Using the transformation (10), this condition becomes

$$\zeta = \bigcirc\left(t^{2-\frac{2\beta}{\pi}}\right) \qquad \text{as} \quad t \longrightarrow 0.$$

3.1.2. Behaviour at the separation point ($t = i$). Locally at the separation point, we have a flow around an angle of γ, thus,

$$f \sim a z^{\frac{\pi}{\gamma}} \qquad \text{as} \quad z \longrightarrow i$$

in terms of the variable t, we write this condition as

$$\zeta = \bigcirc\left((t^2+1)^{2-\frac{2\gamma}{\pi}}\right) \qquad \text{as} \quad t \longrightarrow i.$$

We now have determined the local behaviour of the flow near the singularity and the behaviour far upstream, we seek $\zeta(t)$ in the form

$$\zeta(t) = t^{2-\frac{2\beta}{\pi}}(t^2+1)^{2-\frac{2\gamma}{\pi}}\Omega(t). \tag{11}$$

The function $\Omega(t)$ is bounded and continuous on the unit circle and analytic in the interior. The conditions (8) and (9) show that $\Omega(t)$ can be expanded in the form of a Taylor expansion in even powers of t. Hence,

$$e^{\tau - i\theta} = \zeta(t) = t^{2-\frac{2\beta}{\pi}}(t^2+1)^{2-\frac{2\gamma}{\pi}}\exp\left(\sum_{n=1}^{\infty} a_n t^{2n}\right). \tag{12}$$

By choosing all the coefficients a_n to be real, the function (12) satisfies (9). The coefficients a_n and γ have to be determined to satisfy (7) and (8).

We use the notation $t = |t|\,e^{i\sigma}$ so that points on the BC are given by $t = e^{i\sigma}$, $0 < \sigma < \frac{\pi}{2}$. Using (12) we rewrite (7) in the form

$$\exp(2\bar{\tau}) + \frac{\pi}{\alpha}\exp(\bar{\tau})\tan(\sigma)\frac{\partial\bar{\theta}}{\partial\sigma} = 1. \tag{13}$$

Here $\bar{\tau}(\sigma)$ and $\bar{\theta}(\sigma)$ denote the values of τ and θ on the free surface AB. We solve the problem approximately by truncating the infinite series in (13) after N terms. We find the N coefficients a_n and the separation angle γ by collocation. Thus we introduce the $N+1$ mesh points

$$\sigma_I = \frac{\pi}{2(N+2)}\left(I - \frac{1}{2}\right) \qquad I = 1, \ldots, N+1. \tag{14}$$

Using (14) we obtain $[\bar{\tau}(\sigma)]_{\sigma=\sigma_I}$, $[\bar{\theta}(\sigma)]_{\sigma=\sigma_I}$ and $\left[\frac{\partial\bar{\theta}}{\partial\sigma}\right]_{\sigma=\sigma_I}$ in terms of coefficients a_n and the separation angle γ. Thus, we obtain $(N+1)$ nonlinear algebraic equations of $(N+1)$ unknowns $(a_{n,n=1,\ldots,N}, \gamma)$. The resulting system is solved using Newton's method.

Most of the calculations were done and presented with $N = 60$.

4. Discussion of results

The numerical scheme described in section 3 is used to compute solutions for different values of the inclination angle β and several values of the Weber number α.

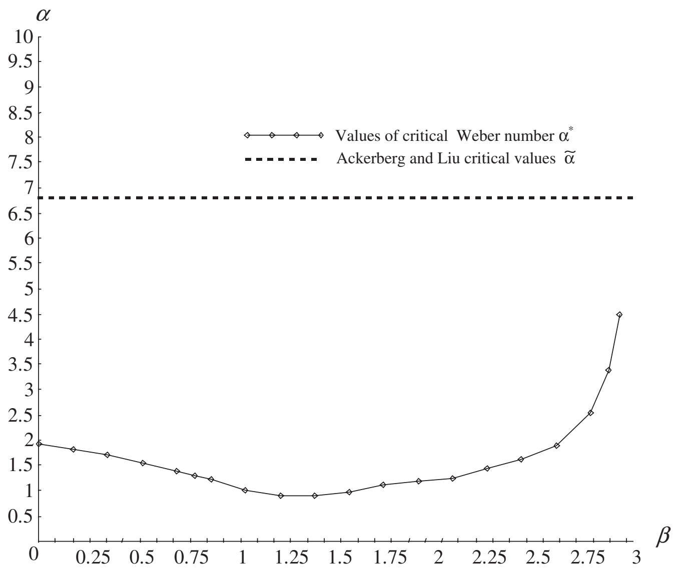


Figure 4. The dividing curve of α^* versus β below which there is no solution. $\tilde{\alpha}$ is the critical value in [1].

Table 1. Some values of the coefficients a_n of the series (12) for several values of the angle β and different values of Weber number α.

β	α	a_1	a_{20}	a_{40}	a_{60}
0	1.916 89	1.059	-0.4444×10^{-3}	-0.7871×10^{-4}	-0.4200×10^{-5}
	30	0.1722	0.2535×10^{-3}	0.3034×10^{-4}	0.3596×10^{-5}
	$\alpha \to \infty$	-0.7942×10^{-9}	0.1539×10^{-15}	0.3917×10^{-16}	-0.3717×10^{-17}
$\frac{\pi}{4}$	1.299 99	0.8876	-0.1198×10^{-3}	-0.1914×10^{-4}	-0.1003×10^{-5}
	30	0.1296	0.1937×10^{-3}	0.2271×10^{-4}	0.2680×10^{-5}
	$\alpha \to \infty$	0.7981×10^{-13}	-3.854×10^{-13}	-1.181×10^{-13}	-0.1030×10^{-15}
$\frac{2\pi}{3}$	1.250 85	0.3995	0.3916×10^{-4}	0.7293×10^{-5}	0.3978×10^{-6}
	30	0.7972×10^{-1}	0.6997×10^{-4}	0.3167×10^{-5}	0.1053×10^{-5}
	$\alpha \to \infty$	0.6264×10^{-8}	0.3123×10^{-15}	0.9756×10^{-16}	0.1275×10^{-16}

4.1. Flow with surface tension effect

When the effect of surface tension is included in the free surface condition, the numerical computation shows that there exists a critical value $\alpha = \alpha^*$ for each value of the inclination angle β (see figure 4), for which there is no solution for $\alpha < \alpha^*$. Accurate solutions for $\alpha \geqslant \alpha^*$ are obtained. As n increases the coefficients a_n decrease rapidly. Table 1 shows some of the coefficients of the series (12) and the corresponding Weber number for different values of β.

The contraction coefficient C is defined as the ratio of the jet width as $x \to \infty$ to the width of the opening.

We note that as the Weber number α decreases, the contraction coefficient C and the angle in the separation point increase. Numerical values of C versus $\frac{1}{\alpha}$ are shown in figure 5.

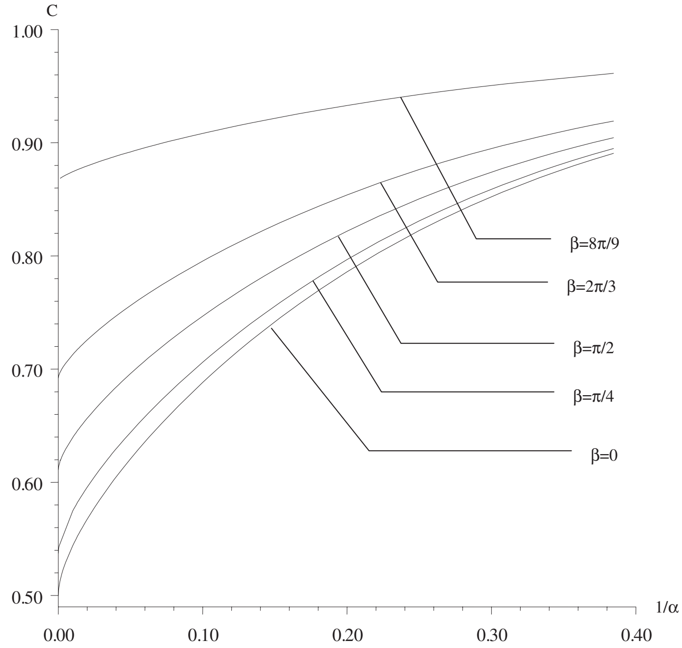


Figure 5. Coefficient of contraction C versus $1/\alpha$.

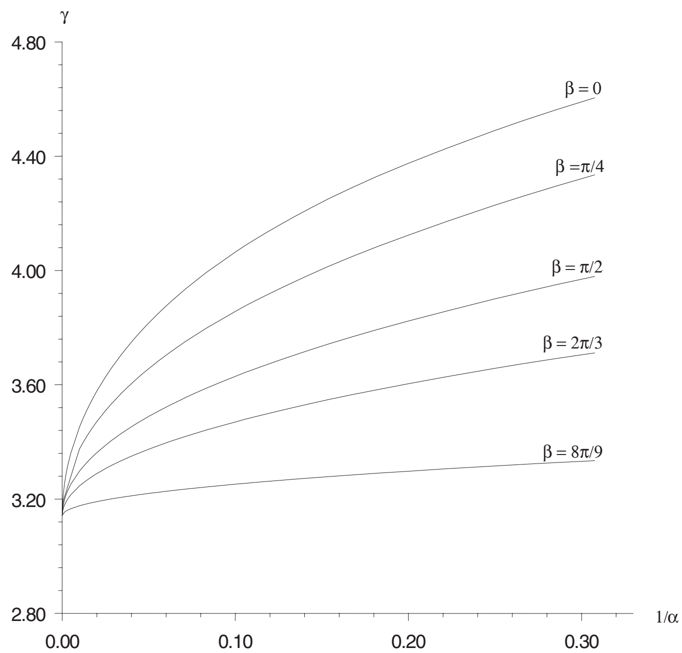


Figure 6. The angle of separation γ versus $1/\alpha$.

In figure 6 we present values of the angle at the separation point between the wall and the free streamlines γ versus $\frac{1}{\alpha}$. It is seen that numerical solutions exist for all $\alpha > \alpha^*$, whereas

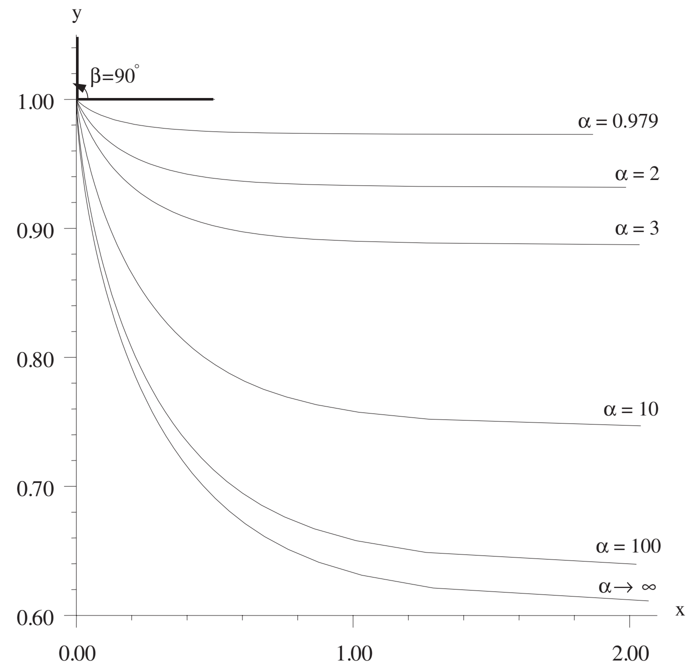


Figure 7. Free streamline shapes for $\beta = \frac{\pi}{2}$ and various Weber numbers.

there is no solution for $\alpha < \alpha^*$. It is also observed that where $\alpha \rightarrow \alpha^*$, the free surface tends to a horizontal line, the contraction coefficient $C \rightarrow 1$ and the angle of separation $\gamma \rightarrow 2\pi - \beta$. For this limiting case, all boundaries are rectilinear, hence, an exact solution can be found via Schwartz–Christoffel transformation [3]. Typical profiles for various Weber numbers of the free surface are presented in figure 7 for $\beta = \frac{\pi}{2}$, figure 8 for $\beta = \frac{2\pi}{3}$ and figure 9 for $\beta = 0$.

For all inclination angles β as $0 \leqslant \beta < \pi$, and different Weber numbers $\alpha \geqslant \tilde{\alpha}$, our results confirm the results of Ackerberg and Liu [1]. These authors solved the problem via the finite difference method and the mesh points were throughout the fluid domain, they could find a solution for all $\alpha \geqslant \tilde{\alpha} = 6.801\,483$. In our procedure mesh points are only needed on the free surface and we computed solutions for $\alpha \leqslant \tilde{\alpha}$. For $\alpha > \tilde{\alpha}$ our results agree with theirs. In figures 10(*a*) and (*b*) we compared the numerical values of the coordinates x and y given in [1] for $\alpha = 32$ and the numerical values computed via our procedure.

4.2. Flow without surface tension

For $\alpha \rightarrow \infty$ and for all inclination angles β, exact analytical solutions can be computed via free stream line theory [4]. We computed these solutions numerically using the procedure described above and our results agree with the theoretical and experimental results given in [2] (figure 11).

For $\beta = \frac{\pi}{2}$, the coefficients $a_n \sim 0$ and the angle $\gamma = 3.1415$, hence the solution is

$$\zeta(t) = t$$

which is the classical Kirchoff solution [3].

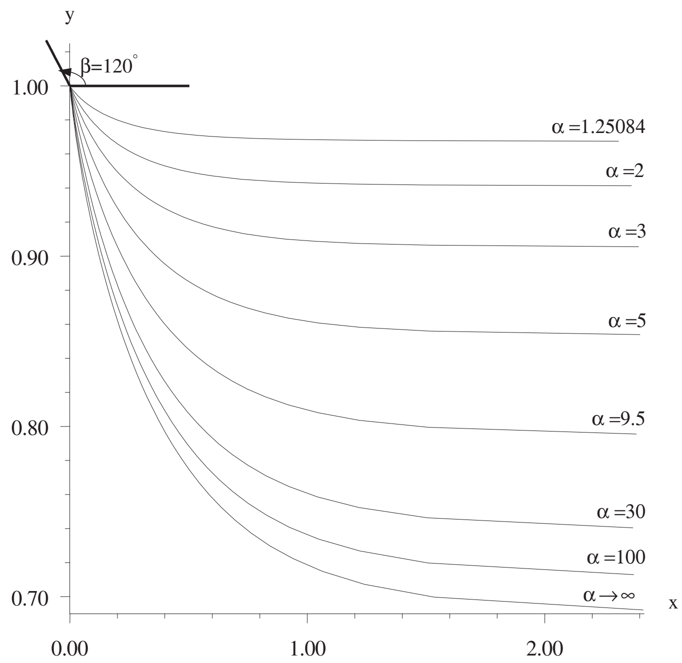


Figure 8. Free streamline shapes for $\beta = \frac{2\pi}{3}$ and various Weber numbers.

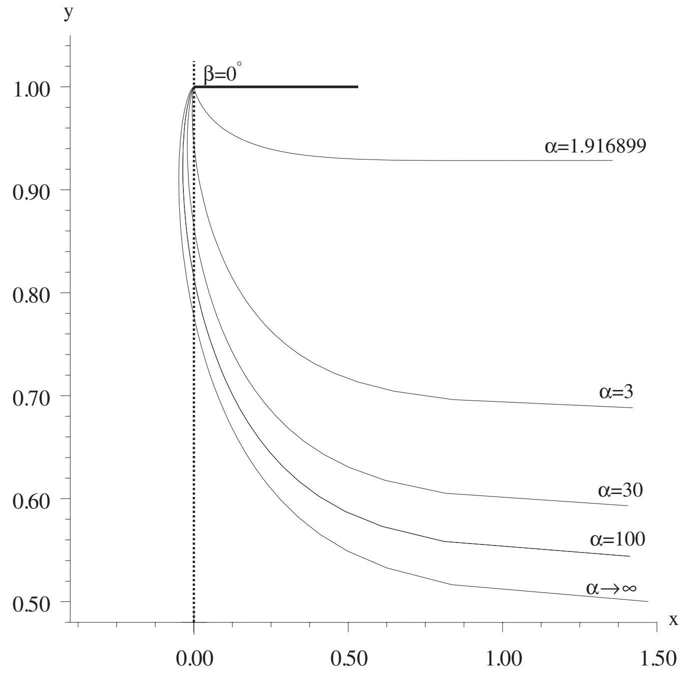
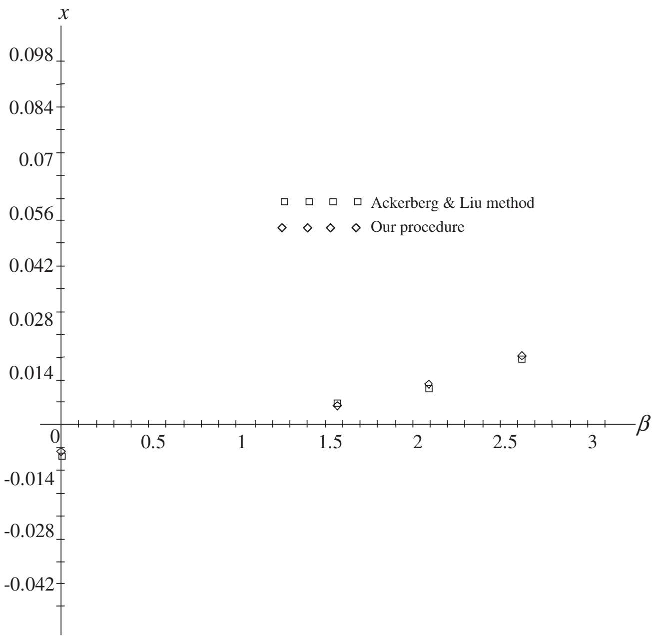
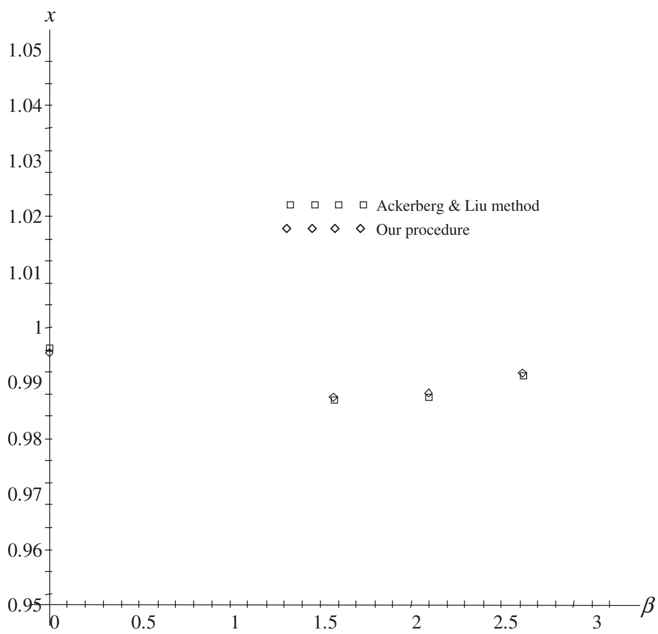


Figure 9. Free streamline shapes for $\beta = 0$ and various Weber numbers.



(a)



(b)

Figure 10. (*a*) Values of the coordinate x for $\sigma = 1.289\,86$ versus β, for the Weber number $\alpha = 32$. (*b*) Values of the coordinate y for $\sigma = 1.289\,86$ versus β, for the Weber number $\alpha = 32$.

For $\alpha \to \infty$ we obtain $C = 0.611$. The comparison of the free streamline shapes for $\beta = \frac{\pi}{2}$ and $\beta = \pi$ obtained using our methods with the theoretical exact solutions are presented in figure 12.

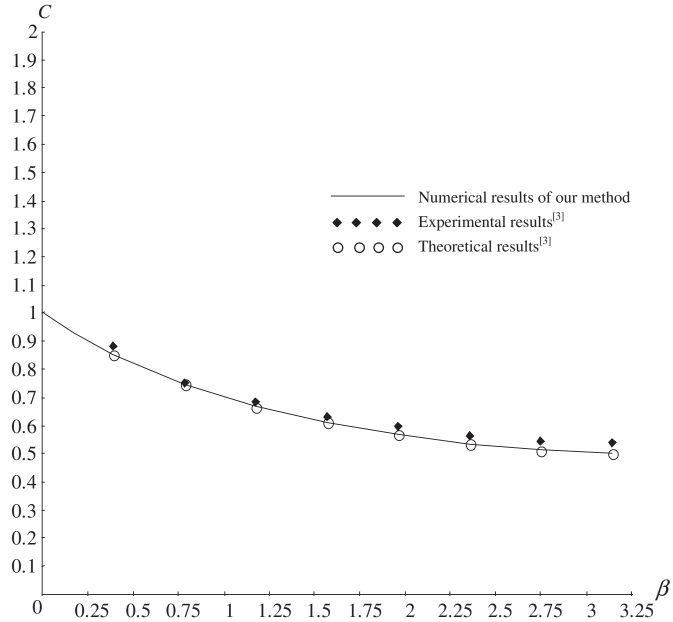


Figure 11. Comparison of contraction coefficients with the theoretical and experimental results given in [2].

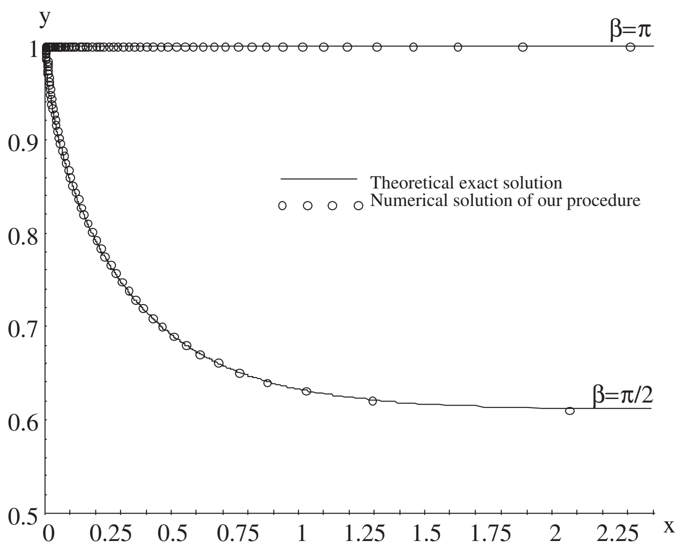


Figure 12. Comparison of the numerical free streamline shape for $\beta = \frac{\pi}{2}$ and $\beta = \pi$ with the exact theoretical results.

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